

Modeling a Spherical Loudspeaker System as Multipole Source

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Introduction

One part of our project “Virtual Gamelan Graz” (VGG) deals with the analysis and re-synthesis of acoustic radiation considering selected Gamelan instruments. Spherical loudspeaker arrays seem to be particularly appropriate for the re-synthesis task. This kind of sound source consists of a solid spherical body, into which individual, separately driven loudspeakers are mounted. In this article, we introduce an analytic model thereof.

Similar to the model of Tarnow [1], we want to model spherical speaker systems, e.g. with the shape of a platonic solid, analytically. Our aim here is not omnidirectional playback, but the playback of Spherical Harmonics, like described in Warusfel [4][5] and Kassakian [6].

The first section shows our analytic model, combining the work of Tarnow [1] and a vibrating spherical cap model, cf. Williams [2] or Meyer [3]. Using the equation of radiation for the multipole source, cf. [2] and Giron [7], we accomplish setting up the acoustic synthesis as a Least-Squares problem. In the last section we show how to use our model to describe the synthesis errors of specific spherical layouts in terms of frequency and distance.

Multipole Source Model

For an analytic description of spherical loudspeaker arrays, we assume a model of the boundary condition for the radial sound particle velocity¹ $v(\varphi, \vartheta)|_{r_0}$ on a sphere with the radius r_0 . We decompose $v(\varphi, \vartheta)|_{r_0}$ into L discrete regions, each one describing the area of a loudspeaker membrane with its own velocity v_l :

$$v(\varphi, \vartheta)|_{r_0} = \sum_{l=1}^L v_l \cdot a_l(\varphi, \vartheta), \quad (1)$$

where the aperture functions $a_l(\varphi, \vartheta)$ can be 1 or 0, and do not overlap, i.e. $\iint a_i(\varphi, \vartheta) a_j(\varphi, \vartheta) d\varphi d\vartheta = 0, \forall i \neq j$:

$$a_l(\varphi, \vartheta) = \begin{cases} 1 & \text{at } l^{\text{th}} \text{ loudspeaker,} \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

Eventually, the residual region $\tilde{a}(\varphi, \vartheta) = 1 - \sum_l a_l(\varphi, \vartheta)$ describes solid parts of the array, where $v = 0$. At first, let us consider an aperture function $\hat{a}(\vartheta)$ of a polar cap with aperture angle² α :

$$\hat{a}(\vartheta) = 1 - u(\vartheta - \alpha/2) \xleftrightarrow{\text{SHT}} \hat{A}_n, \quad (3)$$

¹All relations hold for the frequency domain at ω . We skipped the frequency variable ω in the equations for better readability.

²The unit step function $u(x)$ equals 0 for $x < 0$, and 1 for $x \geq 0$.

and the n^{th} component of its Spherical Harmonic transform \hat{A}_n . We calculate \hat{A}_n by utilizing the Legendre Polynomials $P_n(x)$, cf. [2], [3]:

$$\hat{A}_n = \begin{cases} \cos\left(\frac{\alpha}{2}\right) P_n\left[\cos\left(\frac{\alpha}{2}\right)\right] - P_{n-1}\left[\cos\left(\frac{\alpha}{2}\right)\right], & n > 0 \\ 1 - \cos\left(\frac{\alpha}{2}\right), & n = 0. \end{cases} \quad (4)$$

Regard the work of Meyer [3] as proof of the consistency of this loudspeaker model. By spherical convolution (cf. Yeo [8]) of $\hat{a}(\vartheta)$ with the Dirac distributions³:

$$\delta(\varphi - \varphi_l) \cdot \delta(\vartheta - \vartheta_l) \xleftrightarrow{\text{SHT}} Y_{nm}^*(\varphi_l, \vartheta_l), \quad (5)$$

we move the membrane model $\hat{a}(\vartheta)$ to the locations (φ_l, ϑ_l) of the loudspeakers on the spherical array, cf. Fig. 1. In the Spherical Harmonic domain, isotropic convolution is transformed to a multiplication, cf. [8]. Referring to Eq. 1, we are able to describe the boundary condition in Spherical Harmonics:

$$V_{nm}|_{r_0} = \sum_{l=1}^L v_l \cdot \hat{A}_n \cdot Y_{nm}^*(\varphi_l, \vartheta_l). \quad (6)$$

Inserting Eq. 6 into the equation of radiation for the

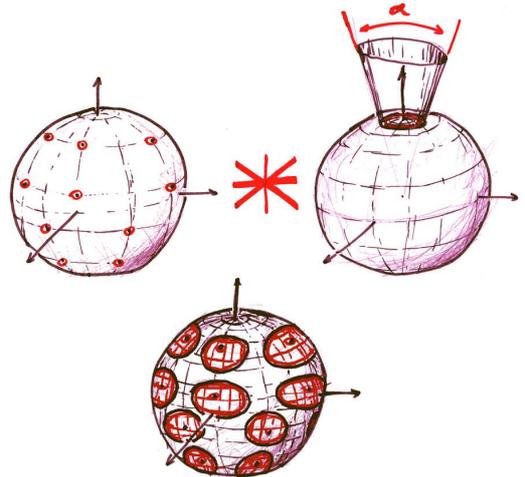


Figure 1: Our loudspeaker array model is a set of boundary conditions for the membrane velocities. We model the speaker vibration as spherical north pole cap with cone angle $\alpha \leq \alpha_{\text{max}}$, moved to its positions by convolution, and weighted by its velocity.

multipole source, cf. Williams [2] and Giron [7], we may

³We use linear indices for the Spherical Harmonics $nm = n^2 + n + 1 + m$ to keep the notation short.

express the sound pressure of the Spherical Harmonic nm of our array model as:

$$Sp_{nm}(kr, kr_0) = i\rho_0 c \frac{h_n^{(2)}(kr)}{h_n^{(2)}(kr_0)} \sum_{l=1}^L v_l \cdot \hat{A}_n \cdot Y_{nm}^*(\varphi_l, \vartheta_l), \quad (7)$$

wherein $i = \sqrt{-1}$, ρ_0 is the sound impedance of the air, c the sound velocity, $k = \frac{\omega}{c}$ is the wave number, r_0 the array radius, $r > r_0$ the radius in space, $h_n^{(2)}(x)$ the spherical Hankel function for radiation, $h_n^{(2)}(x)$ its derivative.

Radiation Synthesis

At this point, we are able to control the radiation by adjusting the loudspeaker velocities v_l in Eq. 7. Suppose, we are given the array radius, hence kr_0 . For the synthesis of the Spherical Harmonic nm at a chosen target argument kr , we now face the Least-Squares problem⁴:

$$\min_{\vec{v}} \sum_{nm'=1}^{(N+1)^2} \|Sp_{nm'}(kr, kr_0) - \delta_{nm}\|_2^2. \quad (8)$$

Its solution provides a vector of suitable velocities $\vec{v} = [v_1, \dots, v_L]^t$. Below, we replace \vec{v} by the extended notation $\vec{v}_{nm}^{(kr, kr_0)}$ to indicate the dependency of the solution on nm and the choice of (kr, kr_0) . Note that we can only control Spherical Harmonics up to the order N , bounded by $N \leq \sqrt{L} - 1$.

Area of Operation

Despite the small Least-Squares errors for Spherical Harmonics up to order $n \leq N$, substantial errors arise due to *aliasing* for Spherical Harmonic orders $n > N$. Nevertheless, because the radial propagation in Eq. 7 suppresses higher orders $n > 2\sqrt{kr_0} - 1$, we get consistent operation under certain circumstances. As a simple criterion, we may require the error measure:

$$\sigma_e^2 = \sum_{nm=0}^{(N+1)^2} \left[\sum_{nm'=0}^{\infty} \left\| Sp_{nm'}(kr, kr_0) \Big|_{\vec{v}_{nm}^{(kr, kr_0)}} - \delta_{nm} \right\|_2^2 \right] \quad (9)$$

to be bounded $\sigma_e^2 < -3\text{dB}$. Here, $\vec{v}_{nm}^{(kr, kr_0)}$ denotes the velocity vector solving the Least-Squares problem in Eq. 8.

Example: Platonic Loudspeaker Systems

Finally, we want to assess the synthesis errors considering platonic loudspeaker layouts. Fig. 2 shows plots of the $\sigma_e^2 = -3\text{dB}$ contour on the corresponding error surfaces. For each layout, the membrane aperture was chosen to be $\alpha = 0.5\alpha_{\max}$, α_{\max} describing the maximum non-overlapping aperture. Note that the icosahedron with 20 faces is the only layout capable of synthesis up to $N = 3 \leq \sqrt{20} - 1$. In this constellation, the bounds are $kr_0 < 2.8$ and $r/r_0 > 2.3$, i.e. given the array radius $r_0 = 0.1\text{m}$, the icosahedral array meets the error target for frequencies $f < 1.5\text{kHz}$ and distances $r > 0.23\text{m}$.

⁴The discrete Dirac delta distribution δ_{nm} equals 1 at $nm' = nm$, and 0 otherwise.

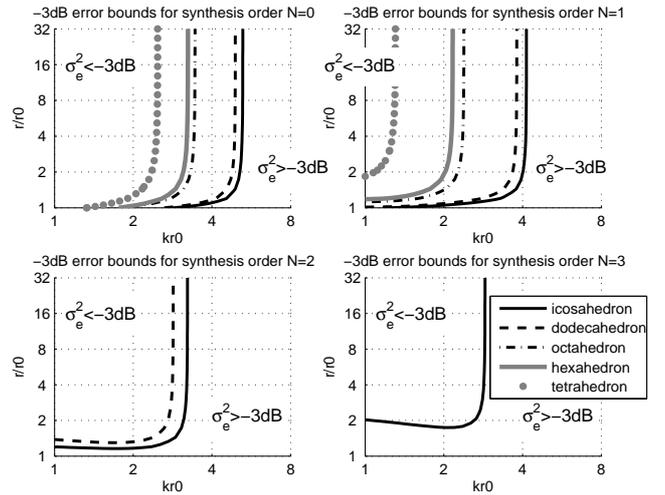


Figure 2: -3dB contours of the Spherical Harmonic synthesis error with platonic spherical arrays.

Conclusions

We have developed an analytical model of spherical loudspeaker arrays dedicated to the synthesis of Spherical Harmonic radiation. Our model turns out a very useful tool, as it can be used to determine the capabilities of spherical loudspeaker array designs.

Acknowledgements

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