Modeling a Spherical Loudspeaker System as Multipole Source

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Introduction

One part of our project “Virtual Gamelan Graz” (VGG) deals with the analysis and re-synthesis of acoustic radiation considering selected Gamelan instruments. Spherical loudspeaker arrays seem to be particularly appropriate for the re-synthesis task. This kind of sound source consists of a solid spherical body, into which individual, seperately driven loudspeakers are mounted. In this article, we introduce an analytic model thereof.

Similar to the model of Tarnow [1], we want to model spherical speaker systems, e.g. with the shape of a platonic solid, analytically. Our aim here is not omnidirectional playback, but the playback of Spherical Harmonics, like described in Warusfel [4][5] and Kassakian [6].

Multipole Source Model

For an analytic description of spherical loudspeaker arrays, we assume a model of the boundary condition for the radial sound particle velocity\(^1\) \(v(\varphi, \vartheta)\)\(_{r_0}\) on a sphere with the radius \(r_0\). We decompose \(v(\varphi, \vartheta)\)\(_{r_0}\) into \(L\) discrete regions, each one describing the area of a loudspeaker membrane with its own velocity \(v_l\):

\[
v(\varphi, \vartheta)\big|_{r_0} = \sum_{l=1}^{L} v_l \cdot a_l(\varphi, \vartheta),
\]

where the aperture functions \(a_l(\varphi, \vartheta)\) can be 1 or 0, and do not overlap, i.e. \(\int a_i(\varphi, \vartheta) a_j(\varphi, \vartheta) \, d\varphi d\vartheta = 0\), \(\forall l \neq j\):

\[
a_l(\varphi, \vartheta) = \begin{cases} 
1 & \text{at } l^{th} \text{ loudspeaker,} \\
0 & \text{otherwise.}
\end{cases}
\]

Eventually, the residual region \(\hat{a}(\varphi, \vartheta) = 1 - \sum_l a_l(\varphi, \vartheta)\) describes solid parts of the array, where \(v = 0\). At first, let us consider an aperture function \(\hat{a}(\vartheta)\) of a polar cap with aperture angle\(^2\) \(\alpha\):

\[
\hat{a}(\vartheta) = 1 - u(\vartheta - \alpha/2) \quad \rightarrow \quad \hat{A}_\alpha,
\]

1^All relations hold for the frequency domain at \(\omega\). We skipped the frequency variable \(\omega\) in the equations for better readability.

2^The unit step function \(u(x)\) equals 0 for \(x < 0\), and 1 for \(x \geq 0\).

\[
\hat{A}_n = \begin{cases} 
\cos \left( \frac{\vartheta}{2} \right) P_n \left[ \cos \left( \frac{\vartheta}{2} \right) \right] - P_{n-1} \left[ \cos \left( \frac{\vartheta}{2} \right) \right], & n > 0 \\
1 - \cos \left( \frac{\vartheta}{2} \right), & n = 0.
\end{cases}
\]

\[
\delta(\varphi - \varphi_l) \cdot \delta(\vartheta - \vartheta_l) \rightarrow Y^\ast_{nm}(\varphi_l, \vartheta_l),
\]

\[
V_{nm}|_{r_0} = \sum_{l=1}^{L} v_l \cdot \hat{A}_n \cdot Y^\ast_{nm}(\varphi_l, \vartheta_l).
\]

Inserting Eq. 6 into the equation of radiation for the multipole source, cf. Williams [2] and Giron [7], we may
express the sound pressure of the Spherical Harmonic $nm$ of our array model as:

$$ S_{pnm}(kr, kr_0) = i \rho_0 c h_n^{(2)}(kr) \sum_{l=1}^{L} \vartheta_l \cdot \hat{A}_n \cdot Y_{nm}^*(\varphi_l, \theta_l), $$

(7)

wherein $i = \sqrt{-1}$, $\rho_0$ is the sound impedance of the air, $c$ the sound velocity, $k = \frac{\omega}{c}$ is the wave number, $r_0$ the array radius, $r > r_0$ the radius in space, $h_n^{(2)}(x)$ the spherical Hankel function for radiation, $h_n^{(2)}(x)$ its derivative.

**Radiation Synthesis**

At this point, we are able to control the radiation by adjusting the loudspeaker velocities $\vartheta_l$ in Eq. 7. Suppose, we are given the array radius, hence $kr_0$. For the synthesis of the Spherical Harmonic $nm$ at a chosen target argument $kr$, we now face the Least-Squares problem$^3$:

$$ \min_{\vec{v}} \sum_{nm=1}^{(N+1)^2} \| S_{pnm}(kr, kr_0) - \delta_{nm} \|_2^2. $$

(8)

Its solution provides a vector of suitable velocities $\vec{v} = [\vartheta_1, \ldots, \vartheta_L]^T$. Below, we replace $\vec{v}$ by the extended notation $\vec{v}_{nm}^{(kr, kr_0)}$ to indicate the dependency of the solution on $nm$ and the choice of $(kr, kr_0)$. Note that we can only control Spherical Harmonics up to the order $N$, bounded by $N \leq \sqrt{L} - 1$.

**Area of Operation**

Despite the small Least-Squares errors for Spherical Harmonics up to order $n \leq N$, substantial errors arise due to aliasing for Spherical Harmonic orders $n > N$. Nevertheless, because the radial propagation in Eq. 7 suppresses higher orders $n > 2\sqrt{kr_0} - 1$, we get consistent operation under certain circumstances. As a simple criterion, we may require the error measure:

$$ \sigma_e^2 = \sum_{nm=0}^{(N+1)^2} \left[ \sum_{nm=0}^{\infty} \| S_{pnm'}(kr, kr_0) \|_{\delta_{nm}^{(kr, kr_0)}} - \| \delta_{nm} \|_2^2 \right], $$

(9)

to be bounded $\sigma_e^2 < -3$dB. Here, $\delta_{nm}^{(kr, kr_0)}$ denotes the velocity vector solving the Least-Squares problem in Eq. 8.

**Example: Platonic Loudspeaker Systems**

Finally, we want to assess the synthesis errors considering platicon loudspeaker layouts. Fig. 2 shows plots of the $\sigma_e^2 = -3$dB contour on the corresponding error surfaces. For each layout, the membrane aperture was chosen to be $a = 0.5 \sigma_{max}, \sigma_{max}$ describing the maximum non-overlapping aperture. Note that the icosahedron with 20 faces is the only layout capable of synthesis up to $N = 3 \leq \sqrt{20} - 1$. In this constellation, the bounds are $kr_0 < 2.8$ and $r > r_0 > 2.3$, i.e. given the array radius $r_0 = 0.1$m, the icosahedral array meets the error target for frequencies $f < 1.5$kHz and distances $r > 0.23$m.

$^4$The discrete Dirac delta distribution $\delta_{nm}$ equals 1 at $nm' = nm$, and 0 otherwise.

**Conclusions**

We have developed an analytical model of spherical loudspeaker arrays dedicated to the synthesis of Spherical Harmonic radiation. Our model turns out a very useful tool, as it can be used to determine the capabilities of spherical loudspeaker array designs.

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**References**


